Fourier Analysis

Apr 18, 2024

Review.

Thm (Poisson summation formula).

Let $f \in \mathcal{M}(\mathbb{R})$. Assume that $\hat{f} \in \mathcal{M}(\mathbb{R})$. Then

$$\sum_{n \in \mathbb{Z}} f(x+n) = \sum_{n \in \mathbb{Z}} \hat{f}(n) e^{2\pi i n x}.$$

In particular,

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n).$$

Thm. Let $f \in M(\mathbb{R})$.

Suppose that $\hat{f}$ is supported on $I = [-\frac{1}{2}, \frac{1}{2}]$, that is,

$\hat{f}(\xi) = 0$ for all $\xi \in \mathbb{R} \setminus I$.

Then

① f is determined by the values of f at $n \in \mathbb{Z}$. More precisely

$$f(x) = \sum_{n \in \mathbb{Z}} f(n) \cdot \frac{\sin(\pi(x-n))}{\pi(x-n)}$$

② $$\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |f(n)|^2.$$

Pf. Write $g = \hat{f}$. Clearly g is cts. Since g is supported on $I = [-\frac{1}{2}, \frac{1}{2}]$, $g \in M(\mathbb{R})$.

Then

$$\hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)\, e^{-2\pi i \xi x}\, dx$$

$$= \int_{-\infty}^{\infty} \hat{f}(x)\, e^{-2\pi i \xi x}\, dx$$

$$\overset{\text{(Inversion formula)}}{=\!=} f(-\xi).$$

So $\hat{g} \in M(\mathbb{R})$.

Notice that g is supported on $[-\frac{1}{2}, \frac{1}{2}]$.

For $x \in [-\frac{1}{2}, \frac{1}{2}]$,

$$g(x) = \sum_{n \in \mathbb{Z}} g(x+n)$$

$$\overset{\text{(by Poisson Summation formula)}}{=\!=} \sum_{n \in \mathbb{Z}} \hat{g}(n)\, e^{2\pi i n x}$$

$$= \sum_{n \in \mathbb{Z}} f(-n) e^{2\pi i n x}$$

$$= \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n x}$$

That is,

$$g(x) = \sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n x} \quad \text{on } [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Next we apply the inversion formula:

$$f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \hat{f}(\xi) e^{2\pi i \xi x} d\xi$$

(since $\hat{f}$ is supported on $[-\frac{1}{2}, \frac{1}{2}]$)

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} g(\xi) e^{2\pi i \xi x} d\xi$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\sum_{n \in \mathbb{Z}} f(n) e^{-2\pi i n \xi} \right) e^{2\pi i \xi x} \, d\xi$$

$$\overset{(DCT)}{=} \sum_{n \in \mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(n) e^{-2\pi i n \xi} \cdot e^{2\pi i \xi x} \, d\xi$$

$$= \sum_{n \in \mathbb{Z}} \int_{-\frac{1}{2}}^{\frac{1}{2}} f(n) \, e^{2\pi i \xi (x-n)} \, d\xi$$

$$= \sum_{n \in \mathbb{Z}} f(n) \cdot \left. \frac{e^{2\pi i \xi (x-n)}}{2\pi i (x-n)} \right|_{\xi = -\frac{1}{2}}^{\frac{1}{2}}$$

$$= \sum_{n \in \mathbb{Z}} f(n) \, \frac{e^{\pi i (x-n)} - e^{-\pi i (x-n)}}{2i \; \pi (x-n)}$$

$$= \sum_{n \in \mathbb{Z}} f(n) \, \frac{\sin(\pi (x-n))}{\pi (x-n)} .$$

This proves ①.

To prove ②, recall that

$$g(x) = \sum_{n \in \mathbb{Z}} \hat{f}(-n) e^{2\pi i n x}$$

(supported on $[-\frac{1}{2}, \frac{1}{2}]$)

Since g is cts and the RHS converges absolutely, the RHS is the Fourier series of g on $[-\frac{1}{2}, \frac{1}{2}]$.

By Parserval identity,

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |g(x)|^2 dx = \sum_{n \in \mathbb{Z}} |\hat{f}(-n)|^2$$

$$= \sum_{n \in \mathbb{Z}} |f(n)|^2$$

Observe that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} |g(x)|^2 dx = \int_{-\frac{1}{2}}^{\frac{1}{2}} |\hat{f}(x)|^2 dx$$

$$= \int_{-\infty}^{\infty} |\hat{f}(x)|^2 dx$$

($\hat{f}$ is supported on $[-\frac{1}{2}, \frac{1}{2}]$)

$$= \int_{-\infty}^{\infty} |f(x)|^2 dx$$

(by Plancherel formula)

That is,

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \sum_{n \in \mathbb{Z}} |f(n)|^2 .$$

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